Mu’tah University
The Graduate Studies Of Deanship

WIDTH IN SURFACES OF CONSTANT

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A Thesis Submitted to the Deanship of the Graduate Studies in Partial Fulfillment of the Requirements for the Degree of Master of Science Department of Mathematics and Statistics

2012University, Mu’tah

ء الآ ا ةدراولداي ف الرسالة الجامعية لاتُعبر

بالضرورة عن وجهة نظر جامعة مؤتة

بسم الله الرحمن الرحيم

MUTAH UNIVERSITY

Deanship of Graduate Studies

جامعة مؤتة

عمادة الدراسات العليا

نموذج رقم (14)

قرار إجازة رسالة جامعية

تقرر إجازة الرسالة المقدمة من الطالب أحمد خضر الناطور الموسومة بـــ:

Surfaces of Constant Width in R3

استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات.

القسم: الرياضيات والأحصاء.

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DEDICATION

ىل إ أمي الحبيبة وأبي الغالي …

إلى اخواني وأختي وأزواجهم وأولادهم وبناتهم …

إلى أصدقائي الأعزاء …

إلى كل من وقف بجانبي في هذه الرسالة …

ويسرني أن اختم إهدائي إلى الإنسان الذي أنار لي طريق العلم

إلى الدكتور كمال البنوي

أحمد الناطور

ACKNOWLEDGEMENT

I thank god for justice, who created everything we know with precise balance and great accuracy of accounts , and I thank god for good fortune to be Dr . kamal Al – Banawi my supervisor of this thesis , supplicating god to give him and his family health and happiness.

Thanks and respect for all teachers in the department of mathematics and statistics , who taught us and as said ;(who has taught me character , I am a slave to him) .

I thank my family , relatives ,friends , and all those who encouraged me to reach my target.

AHMAD ALNATOOR

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Abstract

SURFACES OF CONSTANT WIDTH IN R^3

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Muʼtah University, 2012

The theory of surfaces of constant width in R^3 is a part of classical differential geometry which deals with convex bodies in R^3.

In this thesis we study the main properties of surfaces of constant width in R^3. We use the idea of support function to derive a parametric formula for such surfaces. Such a parametric formula is used to deduce new results regarding the geometry of surfaces.

الملخص

السطوح ذوات العرض الثابت في R^3

أحمد الناطور

جامعة مؤتة، 2012

إن نظرية السطوح ذوات العرض الثابت جزء من الهندسة التفاضلية الكلاسيكية التي تعنى بالأجسام المحدبة في R^3.
في هذه الأطروحة ندرس الخواص الأساسية للسطوح ذوات العرض الثابت R^3 سنستخدم فكرة الاقتران الداعم لأشتقاق صيغة بارامترية لهذه السطوح. إن هذه الصيغة تستخدم لاستنباط نتائج جديدة تتعلق بهندسة السطوح ذوات العرض الثابت في R^3

Chapter One
Introduction

1.1 Preface

This thesis is concerned with the idea of surfaces of constant width in R^3. We build our work on the previous work regarding curves of constant width in R^2. An obvious example of a surface of constant width in R^3 is the sphere S^2, but there are other examples as we will see.

The theory of constant width has generated a considerable literature, much of which is related to the theory of convex bodies and has been developed without differentiability assumptions, as described by, for example, Eggleston [5] or Yaglom and Boltyanskii [11]. However, the differential geometry of curves of constant width started in 1778 when Euler introduced his noncircular plane curves of constant width, *ovals*. His idea is discussed in [10]. It was A.P.Mellish who studied the idea again and generalized it to surfaces and hypersurfaces of constant width [8]. Recent work on the concept of constant width has been done by K.Al-Banawi [1,2,3]. More recently, Al-Banawi and Al-Btoush calculated the area of an oval using the support function [4].

In this thesis we take S as a smooth compact connected surface in R^3 without boundary, so S is closed. If the Gaussian curvature K of S is every where positive, then S is diffeomorphic to S^2 and called an *ovaloid*. If T is a tangent plane of the ovaloid S at a point p, then there is exactly one other tangent plane T^* of S at some other point p^* that is parallel to T. The perpendicular distance $a(p)$ between T and T^* is called the *width* of S at p (or p^*). The points p and p^* are said to be opposite to one another. An ovaloid S is said to be of constant width a if a is independent of p. We develop our study using the *support* function. The support function is the function that measures the perpendicular distance from the tangent plane at a point on the surface to the origin [1,5,11]. Many results concerning surfaces of constant width in R^3 are simply proved using the support function. Also new formulas concerning curvature, focal points and calculations of the support function itself are established.

1.2 Aims and objectives

The main aim of this thesis is to deduce new results concerning the geometry of *surfaces of constant width* in R^3. This aim is built on the idea of using the support function to derive a parametric formula for a surface of constant width in R^3. New formulas concerning curvature, focal points and calculation of the support function itself are established. In general,

the singularity of an ovaloid will be discussed. Examples are built to support the aims of this thesis.

1.3 Summary of thesis

This thesis consists of four chapters including this chapter, the introduction.

Chapter **2** deals with the main concepts needed to understand this work. The main content of Chapter 2 is some differential geometry concerning surfaces in R^3.

A brief description of regular curves in R^2 and regular surfaces in R^3 is introduced. Then coordinate patches and surfaces in R^3 are defined from a topological point of view. The geometry of surfaces in R^3 is described using different types of curvature. The concepts of the first and second fundamental matrices are introduced as a first step in the explanation of the idea of principal curvatures. Then the Gaussian, normal and mean curvatures are defined.

Chapter **3** introduces the concept of an ovaloid , then the concept of a surface of constant width in R^3 as a base for the research in this thesis. Chapter 3 summarizes the main results deduced regarding rotational surfaces of constant width in R^3. Our results are supported by different examples. Chapter 3 introduces the support function as a tool to study surfaces of constant width in R^3, where the interaction between them is explained.

Chapter **4** contains the discussion regarding the singularity of surfaces of constant width in R^3, which is a wide area of research. We introduce new results and keep in mind that many others can be deduced for further work.

Chapter Two
Preliminaries

Most of the material in this chapter is borrowed from O'Neill [9].

2.1 Regular parametrizations

Definition 2.1.1. [9] A curve C in R^2 is a differentiable function

f : where I is an open interval.

Thus, if t is the parameter of f , $t \in I$, then we write f as

$$f(t) = (f_1(t), f_2(t))$$

where f_1 and f_2 are differentiable real valued functions defined on I . We say that f is a parametrization of C or C is the trace of f on I .

Definition 2.1.2. [9] A curve C in R^2 with parametrization

f : is *regular* provided that $f'(t) \neq \vec{0}$ for all $t \in I$. If $f'(t_1) = \vec{0}$ for some $t_1 \in I$, then $f(t_1)$ is a *singular* point of C .

Let U be an open set in R^2 where R^2 has the standard topology. We

define the map F : R^3 by assuming that $(u,v) \in U$ and

$$F(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

Definition 2.1.3. [9] The map F : R^3 is *a regular parametrization* if F_u and F_v both exist on U and $F_u \times F_v \neq \vec{0}$ for all $(u,v) \in U$.

Example 2.1.4. The map F : R^3 with $U = \{(u,v) \in R^2 : u^2 + v^2 < 1\}$ and

$$F(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$$

is a regular parametrization. Here U is open in R^2 and

$$F_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right),$$

$$F_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$$

both exist with $F_u \times F_v = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right) \neq \vec{0}$ on U.

Example 2.1.5. The map F : R^3 with
$U = \{(u,v) (R^{\dagger}(2): u > 0, v > 0\}$ and

$$F(u,v) = (u^2, v^2, uv)$$

is a regular parametrization. Here U is open in R^2 and

$$F_u = (2u, 0, v),$$

$$F_v = (0, 2v, u)$$

both exist with $F_u \times F_v = (-2v^2, -2u^2, 4uv) \neq \vec{0}$ on U.

2.2 Coordinate patches

Definition 2.2.1. [9] The map F : R^3 is a *coordinate patch* if U is open in R^2, F is a regular parametrization and F is a homeomorphism onto its image.

Example 2.2.2. The map in Example 2.1.4 is a coordinate patch. It is one to one and continuous on U. Also the map F^{-1}: $F(U)(U$ defined by

$$F^{-1}(x, y, z) = (x, y)$$

with is continuous on U.

Example 2.2.3. The map in Example 2.1.5 is a coordinate patch. It is one to one and continuous on U . Also the map $F^{-1}: F(U) \to U$ defined by

$$F^{-1}(x,y,z) = (\sqrt{x}, \sqrt{y})$$

with is continuous on U .

2.3 Surfaces in R^3

Definition 2.3.1. [9] A *surface* S in R^3 is a subset of R^3 such that every point of S is covered by a coordinate patch. The set of coordinate patches covering S is called an *atlas*.

Note. If S is covered by one coordinate patch, then S is called a *simple* surface.

Example 2.3.2. The sphere S^2 is a surface. It is covered by six patches similar to the one in Example 2.1.4.

Example 2.3.3. The cylinder which is defined by

$$F(u,v) = (\cos u, \sin u, v), u \in (0,2\pi), v \in R$$

is a surface. It is covered by two patches.

Theorem 2.3.4. Let $f(t) = (x(t), y(t)),\quad y(t) > 0$, $t \in (a,b)$, be a regular curve in R^2 . Assume that f is rotated about the x -axis. Then the result of rotation is a surface.

Proof. The result of rotation is

.

Take $U = (a,b) \times (0,2\pi)$, which is open in R^2.

Now

$$F_u = (x'(u), y'(u)\cos v, y'(u)\sin v),$$

$$F_v = (0, -y(u)\sin v, y(u)\cos v)$$

both exist on U . Moreover,

$F_u \times F_v = y(u)(y'(u), -x'(u)\cos v, -x'(u)\sin v) \neq \vec{0}$ on U .

So F is a regular parametrization. Also F is one to one and continuous on U .

Now the map defined by

$$F^{-1}(p,q,k) = [(x)]^{-1}(p), \tan^{-1}\left(\frac{k}{\square}\right)\Big)$$

is continuous on $F(U)$. So F is a coordinate patch. Two patches are needed to cover the object. Thus, it is a surface.

Theorem 2.3.5. [9] Let h be a differentiable real valued function defined on R^3 and $c \in R$. Let S be the nonempty subset of R^3 defined by $h(x,y,z) = c$. If $dh(p) \neq 0$ for all $p \in S$, then S is a surface.

Proof. Let p be an arbitrary point on S, we have to prove that there exist a coordinate patch covering p.

Since

$$dh(p) = h_x dx + h_y dy + h_z dz \neq 0$$

then at least one of these partial derivatives is not zero say $h_z(p) \neq 0$.

In this case, the implicit function theorem says that near p the equation $h(x,y,z) = c$ can be solved for z. More precisely, it asserts that there is a differentiable real valued function defined on a neighborhood D of $p = (p_1, p_2)$ such that

1) For all $(u,v) \in D$ the point $(u, v, h(u,v))$ lies in *S*, that is $F(u,v,h(u,v)) = c$.

2) The points of the form $(u,v,h(u,v))$ with $(u,v) \in D$ fill a neighborhood of p in S.

It follows immediately that the patch F : R^3 such that $F(u,v) = (u,v,h(u,v))$ covers p. ✧

Example 2.3.6. The set $S = \{(x,y,z) \in R^3 : x^2 + y^2 + z^2 = 1\}$ is a surface. For, let

$$h(x,y,z) = x^2 + y^2 + z^2 - 1.$$

Then

$dh(x,y,z) = 2x dx + 2y dy + 2z dz \neq 0$

for all $(x,y,z) \in S$.

Example 2.3.7. The set $S = \{(x,y,z) \in R^3 : z = x^2 + y^2\}$ is a surface. For, let

$$h(x,y,z) = z - x^2 - y^2.$$

Then

$dh = dz - 2xdx - 2ydy \neq 0$

for all $(x,y,z) \in S$.

Example 2.3.8. The set $S = \{(x,y,z) \in R^3 : z^2 = x^2 + y^2\}$ is not a surface. For, let

$$h(x,y,z) = z^2 - x^2 - y^2.$$

Then

$$dh = 2zdz - 2xdx - 2ydy .$$

Now $dh = 0$ if $(x,y,z) = 0 \in$ (□ .

Definition 2.3.9. [7,9] A surface S with parametrization $F(u,v)$ is *ruled* if F can be written as

$$F(u,v) = f(u) + vg(u).$$

Remark. [7] The curves f and g are called respectively, the *base* and the *director* of S .

Example 2.3.10. The surface defined by

$$F(u,v) = (\cos u, \sin u, v)$$

is ruled with $f(u) = (\cos u, \sin u, 0)$ and $g(u) = (0,0,1)$.

Example 2.3.11. The surface defined by

$$F(u,v) = (u, v, uv)$$

is ruled with $f(u) = (u, 0, 0)$ and $g(u) = (0,1,u)$.

Example 2.3.12. The parametrization

$$F(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

whose image lies on the unit sphere is not ruled.

2.4 Curvatures and convexity

We start with F as a coordinate patch on S . Then we assume that f is a regular curve whose image lies on S . If t is the parameter of f and (u,v) is a local coordinate system of F , then $f(t) = F(u(t), v(t))$ for some smooth real valued functions u and v .

Now

$$f' = F_u u' + F_v v'.$$

Thus,

$$f' \cdot f' = F_u \cdot F_u (u')^2 + 2F_u \cdot F_v (u'v') + F_v \cdot F_v (v')^2.$$

That is

$$f' \cdot f' = [u' \; v'] \begin{bmatrix} F_u \cdot F_u & F_u \cdot F_v \\ F_u \cdot F_v & F_v \cdot F_v \end{bmatrix} \begin{bmatrix} u' \\ □' \end{bmatrix},$$

which is equivalent to

$$f' \cdot f' = \vec{w}^T A \vec{w}.$$

where $\vec{w} = \begin{bmatrix} u' \\ v' \end{bmatrix}$ and A is the well known *first fundamental* matrix.

A choice for a unit normal on S is

$$\hat{n} = \frac{F_u \times F_v}{\|F_u \times F_v\|}.$$

Now

.

Thus,

$$f'' \cdot \hat{n} = (F_{uu} \cdot \hat{n})(u')^2 + (2 F_{uv} \cdot \hat{n})(u'v') + (F_{vv} \cdot \hat{n})(\square')^2.$$

That is

$$f'' \cdot \hat{n} = [u' \ \ v'] \begin{bmatrix} F_{uu} \cdot \hat{n} & F_{uv} \cdot \hat{n} \\ F_{uv} \cdot \hat{n} & F_{vv} \cdot \hat{n} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix},$$

which is equivalent to

$f'' \cdot \hat{n} = \vec{w}^T B \vec{w}$

where B is the well known *second fundamental* matrix.

Definition 2.4.1. [9] The *principal* curvatures of S, denoted by k_1 and k_2, are the eigenvalues of the matrix $A^{-1}B$.

Definition 2.4.2. [9] The *Gaussian* curvature of S is $K = k_1 k_2$.

Definition 2.4.3. [9] The *mean* curvature of S is $M = \dfrac{k_1 + k_2}{2}$.

Definition 2.4.4. [9] The *normal* curvature of is

$k_{\hat{n}} = k_1 \cos^2\theta + k_2 \sin^2\theta$

where θ is the angle between $\hat{n}$ and the principal normal.

Theorem 2.4.5. [9] The Gaussian curvature of S is $K = \dfrac{\det(B)}{\det(A)}$.

Proof. Recall that the determinant of a matrix is the product of its eigenvalues. Thus

$$K = k_1 k_2 = \det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{\det(B)}{\det(A)}.$$

Theorem 2.4.6. [9] The characteristic equation of $A^{-1}B$ is

$\lambda^2 - 2\lambda M + K = 0$.

Proof. The eigenvalues of $A^{-1}B$ are $\lambda = k_1, \lambda = k_2$.
Thus,

,

or

$$\lambda^2 - (k_1 + k_2)\lambda + k_1 k_2 = 0,$$

or

$$\lambda^2 - 2M\lambda + K = 0. \qquad \diamond$$

Definition 2.4.7. [9] A point p on a surface S is called an umbilic point if $k_1(p) = k_2(p)$.

Definition 2.4.8. [9] A surface S is *convex* if the Gaussian curvature K is strictly positive at every point of S.

Example 2.4.9. For the plane $F(u,v) = (u, v, 0)$,

$$A^{-1}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So the principal curvatures of S are $k_1 = 0, k_2 = 0$.
The Gaussian curvature is $K = k_1 k_2 = 0$,
hence the surface S with parametrization F is not convex.
The mean curvature of S is $M = \dfrac{k_1 + k_2}{2} = 0$.
The normal curvature is $k_{\hat{n}} = k_1 \cos^2\theta + k_2 \sin^2\theta = 0$.
Every point on the plane is umbilic.

Example 2.4.10. For the parameterization

$$F(u,v) = (3\cos u \sin v, 3\sin u \sin v, 3\cos v),$$

$$A^{-1}B = \begin{bmatrix} \frac{1}{9\sin^2 v} & 0 \\ 0 & \frac{1}{9} \end{bmatrix}\begin{bmatrix} 3\sin^2 v & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

So the principal curvatures of S are $k_1 = \frac{1}{3}$, $k_2 = \frac{1}{3}$.
The Gaussian curvature is $= k_1 k_2 = \frac{1}{9}$,
hence the surface S with parametrization F is convex.
The mean curvature of S is $M = \dfrac{k_1 + k_2}{2} = \dfrac{1}{3}$.
The normal curvature of S is $k_{\hat{n}} = \frac{1}{3}\cos^2\theta + \frac{1}{3}\sin^2\theta = \frac{1}{3}$.
Every point on the sphere is umbilic.

Example 2.4.11. For the parameterization
$F(u,v) = (3\cos u, 3\sin u, v)$.

$$A^{-1}B = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} & 0 \\ 0 & 0 \end{bmatrix}.$$

So the principal curvatures of S are $k_1 = 0, k_2 = \frac{-1}{3}$.

The Gaussian curvature is ,
hence the surface S with parametrization F is not convex.

The mean curvature is
The normal curvature is $k_{\hat{n}} = k_1 \cos^2\theta + k_2 \sin^2\theta$

$$= \frac{-1}{3}\sin^2 \square .$$

There are no umbilic points.

Chapter Three
The Theory of Constant Width

3.1 Surfaces of constant width

Suppose that S is a smooth compact connected surface in R^3 without boundary, so S is closed. If the Gaussian curvature K of S is every where positive, then S is diffeomorphic to the sphere S^2 and called an *ovaloid*. If T is a tangent plane of the ovaloid S at a point p, then there is exactly one other tangent plane T^* of S at some other point p^* that is parallel to T. The perpendicular distance $a(p)$ between T and T^* is

called the *width* of S at p (or p^*). The points p and p^* are said to be opposite to one another.

Definition 3.1.1. [8] An ovaloid S is said to be of *constant width* a if a is independent of p .

Theorem 3.1.2.[8] For a surface S in R^3, The following statements are equivalent:

1) The surface S is of constant width a .
2) The affine normal lines at opposite points p and p^* of S coincide.
3) Opposite points p and p^* on S have the same set of focal points.
4) If the corresponding radii of curvature are ρ_1 , ρ_2 at p and ρ_1^* , ρ_2^* at p^* then $\rho_1^* + \rho_1 = = \rho_2^* + \rho_2$.

Note. The cross section of the surface S through p and p^* is a curve of constant width a . For such a curve of constant width the distance b between the normal lines is identically zero. Thus, for the above surface $b(u,v) = 0$ for all (u,v). Hence the affine normal lines coincide.

Theorem 3.1.3. Let $f : R \to R^2$ defined by $f(u)=(f_1(u), f_2(u))$ be a curve of constant width a with $f(u^*)$ as an opposite point of $f(u)$. Assume that f is symmetric about a certain line in R^2. If f is rotated about that line of symmetry, then the surface of revolution F is of constant width a with $F(u^*,v)$ being the opposite point of $F(u,v)$ where v is the angle of rotation.

Proof. Since $f(u^*)$ is an opposite point of $f(u)$ then $T_f(u^*) + T_f(u) = \vec{0}$, where $T_f(u)$ is the tangent to f at u . Hence $\frac{df_1}{du}(u^*) = -\frac{df_1}{du}(u)$ and $\frac{df_2}{du}(u^*) = -\frac{df_2}{du}(u)$. Without loss of generality, the rotation by an angle v can be taken as a rotation about the x -axis, $v \in [0, \pi]$. Hence

$$F(u,v) = (f_1(u), f_2(u)\cos v, f_2(u)\sin v).$$

Then

$$F_u = \left(\frac{df_1}{du}, \frac{df_2}{du}\cos v, \frac{df_2}{du}\sin v\right)$$

and

$$F_v = \left(0, -f_2(u)\sin v, f_2(u)\cos v\right),$$

and so

$$\hat{n}(u,v)=\frac{1}{\sqrt{\left(\frac{df_2}{du}\right)^2+\left(\frac{df_1}{du}\right)^2}}\left(\frac{df_2}{du},-\frac{df_1}{du}\cos v,\frac{df_1}{du}\sin v\right)$$

is the normal of F at $F(u,v)$.

Now $\hat{n}(u^{*},v)=-\hat{n}(u,v)$, and so $F(u^{*},v)$ is the opposite point of $F(u,v)$.

Also $d\,(f(u^{*},v),f(u,v))=d\,(f(u^{*}),f(u))=a$, and so F is of constant width a. ✧

Example 3.1.4. Consider the unit circle

$$f(u)=(\cos u,\sin u),\ u(R\ .$$

Of course f is a curve of constant width 2. The surface of revolution F is a sphere and defined by

$$F(u,v)=(\cos u,\sin u\cos v,\sin u\sin v),\ v([0,\text{"}(\text{"}]\ .$$

By Theorem 3.1.3 $F(u,v)$ and $F(u+(,v)$ are opposite points and the surface is of constant width 2.

Now

$$F_u=(-\sin u,\cos u\cos v,\cos u\sin v)$$

and

$$F_v=(0,-\sin u\sin v,\sin u\cos v),$$

and so

$$\hat{n}=\frac{F_u\times F_v}{\|F_u\times F_v\|}=(\cos u,\sin u\cos v,\sin u\sin v).$$

Now let $L(u,v)$ be the affine normal line of F at $F(u,v)$, and let   , then

$$\begin{aligned} q &= F(u,v)+\,[\,n^{*}(u,v), \qquad ((R \\ &= F(u+(,v)+2\hat{n}(u+(,v)-[\,\hat{n}(u+(,v) \\ &= F(u+(,v)+(2-()\hat{n}(u+(,v). \end{aligned}$$

Hence   . Similarly, if   , then   . Thus, the affine normal lines at $F(u,v)$ and $F(u+(,v)$ coincide.

The first and the second fundamental matrices of F respectively are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 u \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2 u \end{bmatrix}.$$

The eigenvalues of $A^{-1}B$ are $k_1 = -1$, $k_2 = -1$ and so if $G_1(u,v)$ and $G_2(u,v)$ are the focal points of the surface based at $F(u,v)$, then

$$G_1(u,v) = F(u,v) + \frac{1}{k_1}\hat{n}(u,v) = \vec{0},$$

$$G_2(u,v) = F(u,v) + \frac{1}{k_2}\hat{n}(u,v) = \vec{0}.$$

Example 3.1.5. To construct a nonspherical ovaloid of constant width 18, consider the oval defined by

, .

The above oval is due to Fillmore [6].

If f is rotated about the x -axis in R^3. Then the result of rotation is

$$F(u,v) = \quad ,$$

$$\left(9\sin u + \sin u \cos 3u - 3\cos u \sin 3u\right)\cos v,$$

where v is the angle of rotation.

By Theorem 3.1.3 $F(u,v)$ and $F(u+(,v)$ are opposite points and the surface is of constant width 18.

If $\gamma(u) = 9 - 8\cos 3u, \beta(u) = 9\sin u + \sin u\cos 3u - 3\cos u \sin 3u$.

Then

$$F_u = \gamma(u)(\sin u, \cos u \cos v, \cos u \sin v)$$

and

$$F_v = \beta(u)(0, \sin v, \cos v).$$

A unit normal of F at $F(u,v)$ is

$$\hat{n}(u,v) = (-\cos u, -\sin u \cos v, -\sin u \sin v).$$

Now let $L(u,v)$ be the affine normal line of F at $F(u,v)$, and let $q(L(u,v)$, then

$$q = F(u,v) + (n^*(u,v), ((R$$

$$= F(u+(,v) + 18n(u+(,v) - (n^{\cdot}(u+{}^{\cdot}(",\square)$$

$$= F(u+(,v) + (18-()\hat{n}(u+(,v).$$

Hence $q(L(u+{}^{\cdot}(",\square)$. Similarly, if $q(L(u+{}^{\cdot}(",\square)$, then $q(L(u,v)$. Thus, the affine normal lines at $F(u,v)$ and $F(u+(,v)$ coincide.

The first and second fundamental matrices of F are respectively

$$A = \begin{bmatrix} \gamma^2(u) & 0 \\ 0 & \beta^2(u) \end{bmatrix} \text{ and } B = \begin{bmatrix} \gamma(u) & 0 \\ 0 & \beta(u)\sin u \end{bmatrix}.$$

The eigenvalues of $A^{-1}B$ are $k_1 = \frac{1}{\gamma(u)}$ and $k_2 = \frac{\sin u}{\beta(u)}$ and so if $G_1(u,v)$ and $G_2(u,v)$ are the focal points of the surface based at $F(u,v)$, then

$$G_1(u,v) = F(u,v) + \frac{1}{k_1}\hat{n}(u,v)$$

$$=$$

$$(9\sin u\cos 3u - 3\cos u\sin[3u)\cos v,]$$

and

$$G_2(u,v) = F(u,v) + \frac{1}{k_1}\,\hat{n}(u,v) = \frac{1}{\sin u}(3\sin 3u, 0, 0),$$

which are the two focal points of the surface based at the point $F(u+(,v)$.

3.2 The support function

The idea of the envelope of tangent planes of a smooth closed convex surface in R^3 can be used to deduce a parametric formula for such a surfaces in R^3 [1]. Suppose that S is a smooth closed convex surface in R^3. Without loss of generality, assume that S contains the origin in its interior. Consider the ray that starts from the origin and makes an angle u with the positive x-axis, $u([0,{}^{\cdot}(^{\cdot}]$. Assume that the projection of the ray on the yz-plane makes an angle v with the positive y-axis, $v([0,2^{\cdot}(^{\cdot}]$. Among the perpendicular planes to that ray, take the one that is tangent to the surface. The point of tangency depends on u and v and we write it as

$$F(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

Let $((u, v)$ be the distance from the origin to the tangent plane. The function (is called the *support* function.
The equation of the tangent plane to S is

. (3.1)

The differentiation of Equation (3.1) with respect to u and v leads respectively to

(3.2)

and

. (3.3)

Solving the above system for x , and z leads to

, (3.4)

(3.5)

and

. (3.6)

Thus, the envelope of the tangent planes is the surface S which is induced by (and defined by $F(u,v)=(x(u,v),y(u,v),z(u,v))$.

Theorem 3.2.1. Let S be a smooth closed convex surface in R^3 as above. Let (be the support function of S with parameters u and v and F the parametrization of S induced by (. If S is of constant width a, then the equation of S can be written as

$$F(u,v)=\left(\frac{a}{2}\cos u+H,\frac{a}{2}\sin u\cos v+G,\frac{a}{2}\sin u\sin v+T\right) \quad (3.7)$$

where

1) The functions , G and T are functions of u and v with

$$H((-u,v+()=H(u,v),$$
$$G((-u,v+()=G(u,v)$$

and

$$T((-u,v+()=T(u,v).$$

2) The support function of S is

$$(=\frac{a}{2}+H\cos u+\sin u\,(G\cos v+T\sin v) \quad (3.8)$$

3) The vector $\hat{n}=(\cos u,\sin u\cos v,\sin u\sin v)$ is a common unit normal for the surface defined by (H , G ,) and S .

Proof. The proof of this theorem is built on the fact that $F((-u,v+()$ is an opposite point of $F(u,v)$. So

$$d(F(u,v),F("(" -u,v+"("))=((u,v)+(("(" -u,v +"(")=a .$$

By differentiation with respect to u and , we have

and

.

Now since

$$x(u,v) - x((-u, v + () = a\cos u \qquad (3.9)$$

(3.10)

and

(3.11)

then the equation of the surface S can be written with components

$$x(u,v) = \frac{a}{2}\cos u + x(u,v) - \frac{a}{2}\cos u, \qquad (3.12)$$

$$y(u,v) = \frac{a}{2}\sin u \cos v + y(u,v) - \frac{a}{2}\sin u \cos v \qquad (3.13)$$

and

$$z(u,v) = \frac{a}{2}\sin u \sin v + z(u,v) - \frac{a}{2}\sin u \sin v. \qquad (3.14)$$

Let , G and T be the smooth functions defined by

$$H_{\square}(u,v) = x(u,v) - \frac{a}{2}\cos u \qquad (3.15)$$

(3.16)

and

.

(3.17)

Then F has the form of Identity(3.7). Also

$H(u,v) - H((- u, v + () =$

$$\left(x(u,v) - x((- u, v + () - \left(\frac{a}{2}\cos u - \frac{a}{2}\cos((- u)\right)\right)$$

$$= a\cos u - \left(\frac{a}{2}\cos u - \frac{a}{2}\cos\left((- u\right)\right)$$

$$= a\cos u - a\cos u = 0 \, .$$

Also

$G(u,v) - G((- u, v + () =$

$$y(u,v) - y((- u, v + () - \left(\frac{a}{2}\sin u\cos v - \frac{a}{2}\sin((- u)\cos(v + ()\right)$$

$$= a\sin u\cos v - \left(\frac{a}{2}\sin u\cos v - \frac{a}{2}\sin((- u)\cos(v + ()\right)$$

$$= a\sin u\cos v - a\sin u\cos v = 0$$

and

$T(u,v) - T((- u, v + () =$

$$z(u,v) - z((- u, v + () - \left(\frac{a}{2}\sin u\cos v - \frac{a}{2}\sin((- u)\cos(v + ()\right)$$

$$= a\sin u\sin v - a\sin u\sin v = 0 \, ,$$

and so (1) is proved.

The surface S is defined either by using the support function, Equations (3.4,3.5,3.6), or by using the functions H, G and T, Identity(3.7). Combining both forms of S leads to

(3.18)

and so (2) is proved.
Also the combination of both forms leads to

$$\cos u(G\cos v + T\sin v) \qquad (3.19)$$

and

(3.20)

The differentiation of Equation(3.18) with respect to u together with Equation (3.19) leads to

$$H_u \cos u + \quad G_u \sin u \cos v + \quad T_u \sin u \sin v = 0 \,.$$

Similarly the differentiation of Equation (3.18) with respect to v together with Equation(3.20) leads to

$$H_v \cos u \; + G_v \sin u \cos v + \quad T_v \sin u \sin v = 0 \,.$$

Thus, the vector $\hat{n} = (\cos u, \sin u \cos v, \sin u \sin □)$ is a unit normal for the

surface defined by (H , G ,). Since

$$< \hat{n}, (-\sin u, \cos u \cos v, \cos u \sin v)> = 0$$

and

$$<\hat{n}, (0, -\sin u \sin v, \sin u \cos □) > = 0 \,.$$

The vector $\hat{n}$ is also a unit normal for the surface S , and so (3) is proved.
✧

Theorem 3.2.2. Let S be a surface of constant width a . Then

$$\frac{k_{\hat{n}}}{K}(p) + \frac{k_{\hat{n}}}{K(p^*)} = a \,,$$

where and $k_{\hat{n}}$ are the Gaussian and normal curvature respectively.
Proof. By the fact that $k_{\hat{n}} = k_1 \cos^2\theta + k_2 \sin^2\theta$ and $K = k_1 k_2$ to get

$$\frac{k_{\hat{n}}}{K}(p)+\frac{k_{\hat{n}}}{K}(p^*) = \frac{k_1\cos^2\theta + k_2\sin^2\theta}{k_1 k_2} + \frac{k_1^*\cos^2\theta + k_2^*\sin^2\theta}{k_1^* k_2^*}$$
$$= \frac{\cos^2\theta}{k_2} + \frac{\sin^2\theta}{k_1} + \frac{\cos^2\theta}{k_2^*} + \frac{\sin^2\theta}{k_1^*}$$
$$= \left(\frac{1}{k_2}+\frac{1}{k_2^*}\right)\cos^2\theta + \left(\frac{1}{k_1}+\frac{1}{k_1^*}\right)\sin^2\theta$$
$$= a(\cos^2\theta + \sin^2\theta) = a\,. \qquad \diamond$$

3.3 A rotational surface of constant width

We state a necessary condition for a surface of constant width to be rotational.

Theorem 3.3.1. Let S be a surface of constant width a in R^3 as above. Then the surface S is the rotation of a curve of constant width a in R^2 about the x-axis iff the support function of S is independent of v.

Proof. Let γ be the support function of the surface S. If $\gamma = \gamma(u)$, then $\gamma_v = 0$ and $\gamma_u = \frac{d\gamma}{du}$. Hence if S is defined by

$$F(u,v) = (x(u,v), y(u,v), z(u,v)),$$

then Equations (3.4, 3.5, 3.6) reduce to

$$x(u,v) = \gamma\cos u - \frac{d\gamma}{du}\sin u \qquad (3.21)$$

$$y(u,v) = \left(\gamma\sin u + \frac{d\gamma}{du}\cos u\right)\cos v \qquad (3.22)$$

and

$$z(u,v) = \left(\gamma\sin u + \frac{d\gamma}{du}\cos u\right)\sin v\,. \qquad (3.23)$$

Let $\beta: [0,2\pi] \to R^+$ be the continuous function defined by

$$\beta(u) = \begin{cases} \gamma(u) & u \in [0,\pi] \\ \gamma(2\pi - u) & u \in [\pi, 2\pi] \end{cases}$$

Consider the curve C parametrized by u and defined by

(3.24)

where $u \in [0,2\pi]$.

The curve C has the function β as its own support function and it is of constant width a, since

if , then $\beta(u) + \beta(u + \pi) = \gamma(u) + \gamma(\pi - u) = a$

and

if $u \in [\pi/2, 2\pi]$, then $\beta(u) + \beta(u + \pi) = \beta(u^{-} + \pi) + \beta(u^{-} + 2\pi)$

$= \gamma(\pi - u^{-}) + \gamma(u^{-}) = a$, where .

There are two ways to describe S as a result of the rotation of C about the *x*-axis. The first way is to restrict the domain of f to $[0, \pi]$. If the part of C left after this restriction is rotated by an angle v about the *x*-axis, which is an axis of symmetry, then with $v \in [0, 2\pi]$, S is the result of this rotation. Geometrically, this implies that S is the rotation of half the curve about the x-axis. In this case, the components of F are written using γ and $\frac{d\gamma}{du}$. The second way is to consider S as the rotation of the whole curve C around the x-axis. Thus, $u \in [0, 2\pi]$ and $v \in [0, \pi]$. In the last case, the components of F are written using β and $\frac{d\beta}{du}$. By Theorem 3.1.3, $F(u + \pi, v)$ is the opposite point of $F(u, v)$.

Now suppose that S is the rotation of a curve C of constant width a in R^2 about the x-axis. Assume that the curve is defined by $f = (f_1, f_2)$ with support function β and a parameter $u, u \in [0, 2\pi]$. Let v be the angle of rotation, $v \in [0, \pi]$. The surface S can also be obtained by the rotation of the part of the curve left after the restriction of the domain of f to $[0, \pi]$ while $v \in [0, 2\pi]$. The substitutions

$$\rho = \gamma, \quad x = f_1, \quad y = f_2 \cos v \text{ and } z = f_2 \sin v$$

in Equations(3.4, 3.5, 3.6) leads to $\gamma_v = 0$, and so the support function of S is independent of v since

$$f_1 = \beta \cos u - \frac{d\beta}{du} \sin u \quad \text{and} \quad f_2 = \beta \sin u + \frac{d\beta}{du} \cos u.$$

The support function of S is defined by

$$\gamma(u,v) = \beta(u).u\in[0,\pi].v\in[0,2\pi] \text{ .} \qquad \diamondsuit$$

The proof of Theorem 3.3.1 tells that a rotational surface can be written using the support function of its rotated curve. Also $F(u+\pi,v)$ is the opposite point of $F(u,v)$. This will help to conclude some results about a rotational surface of constant width similar to the ones of its rotated curve of constant width.

In [1], Al-Banawi proved that for a curve f of constant width in R^2, the parametrization of f is

$$f(u) = \left(\frac{a}{2}\cos u + h(u), \frac{a}{2}\sin u + g(u)\right)$$

where h and g are real valued functions. Also in [1] Al-Banawi proved that $\frac{dh}{du} = 4A(u)\sin u$ and $\frac{dg}{du} = -4A(u)\cos u$ where $A(u)$ is a real valued function.

If S is a rotational surface of constant width a, then the

functions , G and T are related to the functions h and g in the previous Identity in the following way

$$H = h, G = g\cos v \text{ and } T = g\sin v.$$

Hence

(3.25)

Theorem 3.3.2. Let S be a surface of constant width a in R^3. Assume that S is the rotation of a curve of constant width a in R^2 about the *x*-axis as above. Then the two sets of focal points of S are the images of G_1 and G_2 where

$$G_1 = \left(h - \frac{dg}{du}, \left(g + \frac{dh}{du}\right)\cos v, \left(g + \frac{dh}{du}\right)\sin v\right) \qquad (3.26)$$

and

$$G_2 = (h - g\cot u, 0, 0) \qquad (3.27)$$

Proof. The surface is defined by Equation 3.25. Hence

$$F_u = \left(\frac{a}{2} - 4A(u)\right)\left(-\sin u, \cos u \cos v, \cos u \sin v\right)$$

and

$$F_v = \left(\frac{a}{2}\sin u + g\right)\left(0, -\sin v, \cos v\right)$$

where $A(u)$ is a real valued function. A unit normal of F at $F(u,v)$ is

$$\hat{n}(u,v) = \left(-\cos u, -\sin u \cos v, -\sin u \sin v\right).$$

Also

$$F_{uu} = \left(\frac{a}{2} - 4A(u)\right)\left(-\cos u, -\sin u \cos v, -\sin u \sin v\right) - 4\frac{dA}{du}\left(-\sin u, \cos u \cos v, \cos u \sin v\right),$$

$$F_{uv} = \left(\frac{a}{2} - 4A(u)\right)\left(0, \cos u \sin v, -\cos u \cos v\right) + \cos u\left(\frac{a}{2} - 4A(u)\right)\left(0, \sin v, \cos u\right)$$

and

$$F_{vv} = \left(\frac{a}{2}\sin u + g\right)\left(0, -\cos v, -\sin v\right).$$

The first and second fundamental forms of F are respectively

and

$$B = \begin{pmatrix} \frac{a}{2} - 4A(u) & 0 \\ 0 & \sin u\left(\frac{a}{2}\sin u + g\right) \end{pmatrix}.$$

The principal curvatures of F are the eigenvalues of $A^{-1}B$ which are

$$\frac{1}{\frac{a}{2} - 4A(u)} \text{ and } \frac{\sin u}{\frac{a}{2}\sin u + g}.$$

The calculation of $F + \frac{1}{k}n$ will give the forms of the focal points as in the theorem. ✧

The first set of focal points is just the result of the rotation of the curve of focal points of C about the x -axis. The second set of focal points

is just a line segment which lies on the *x*-axis. Also opposite points have the same set of focal points. The principal directions at $F(u,v)$ are $\frac{1}{\frac{a}{2}-4A(u)}F_u(u,v)$ and $\frac{1}{\frac{a}{2}\sin u+g}F_v(u,v)$. They are both parallel to the

corresponding principal directions at . Let ρ_1 , ρ_2 be the principal radii of curvature. Then $\rho_1(u) = \frac{a}{2}-4A(u)$ and

. It is clear that for $i=1,2$,

.

The next example is a surface of constant width in R^3 that is not rotational.

Example 3.3.3. Let $\rho(u,v)= a/2+\cos v$. The support function depends on v , and so the surface S induced by ρ is not rotational. If S is defined by F , then the components of F are

$$x = \frac{a}{2}\cos u+\cos u\cos v,$$

$$y = \frac{a}{2}\sin u\sin v+\sin u\cos^2 v+\csc u\, sin^2 v$$

and

$$z = \frac{a}{2}\sin u\sin v+\sin u\sin v\cos v-\csc u\sin v\cos v.$$

The vector $\hat{n}(u,v)=(\cos u,\sin u\cos v,\sin u\sin v)$ is a unit normal of S at $F(u,v)$. Since $n^{\sim}(\pi-u,v+\pi) = -n^{\sim}(u,v)$, the surface S has $F(\pi-u,v+\pi)$ as opposite to $F(u,v)$. The surface is of constant width a .

Chapter Four
Singularity of Surfaces of Constant width

4.1 Introduction

Let $x(u,v), y(u,v)$ and $z(u,v)$ be as in Equations (3.4, 3.5 , 3.6). We are going to study the coordinates of a convex surface via differentiation.
So

(4.1)

(4.2)

(4.3)

(4.4)

(4.5)

(4.6)

The appearance of $\sin u$ in the denominator means that $u = 0, ($ are excluded.

4.2 Singularity of surfaces

Definition 4.2.1. Let S be a surface with parametrization $F(u,v) = (x(u,v), y(u,v), z(u,v))$ where $x(u,v), y(u,v)$ and $z(u,v)$ as in Equations (3.4, 3.5 , 3.6). A point on F is singular if the matrix

$$E = \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix}$$

has rank less than 2.

Theorem 4.2.2. Let S be a surface with parametrization F as above. Then F has a singular point at (u,v) iff $-((u,v)$ is an eigenvalue of the Hessian matrix

.

Proof. Since F has a singularity at (u,v), then the rank of the matrix E is less than 2, which is equivalent to

$$\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} = \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = 0 .$$

Without loss of generality, substitute $u = \frac{\pi}{2}, v = 0$ in Equations (4.1, 4.2, 4.3, 4.4, 4.5, 4.6). Then we have

= ,

which leads to

.

Now

.

So $-($ is an eigenvalue of the Hessian matrix H .
The converse is no more than running the calculation backward.

Theorem 4.2.3. Let S be a surface of revolution with support function $(=((u)$. Then F has a singularity at (u_1,v_1) if $(^{\prime\prime}(u_{\downarrow}1)+((u_{\downarrow}1)=0$.

Proof. By calculating $(_{\iota}uu$, $(_{\iota}vv$ and $(_{\iota}uv$ and substituting them in the Hessian matrix, we get

$$H=[\blacksquare((^{\prime\prime}\&0@0\&0)]\ .$$

By Theorem 4.2.2, F has a singularity at (u_1,v_1) iff $-(=0$ or $-(=(^{\prime\prime}$.
Now $(=0$ is ignored since $($ is always positive.
Hence $-(=(^{\prime\prime}$, so $(^{\prime\prime}+(=0$. ✧

Theorem 4.2.4. Let S be a surface of revolution with support function

,

where a and b are constant with $b>a>0$, and n is an odd number

$n>1$. Then has a singularity if .

Proof. The parametrization F has a singular point at (u,v) if $-((u,v)$ is an eigenvalue of the Hessian matrix

.

By the evaluation of $(_{\iota}uu$, $(_{\iota}uv$ and $(_{\iota}vv$, we get

$$H = \begin{bmatrix} -an^2\cos nu & 0 \\ 0 & 0 \end{bmatrix},$$

and so the eigenvalues of H are $-an^2\cos nu$ and 0 .

Case 1. If $-(= 0$ then $-(a\cos nu + b) = 0$, which implies that

$\cos nu = \dfrac{-b}{a}$, which is impossible.

Case 2. If $-(= -an^2\cos nu$, then

$$-(a\cos nu + b) = -an^2\cos nu,$$

and so

$$\cos nu = \frac{b}{a(n^2-1)}.$$

But . Thus, $|b/a(n\uparrow 2-1)|(1$

and so

$$b(a(n\uparrow 2-1) \qquad ✧$$

4.3 Singularity of surfaces of constant width

We explain Theorem 4.2.4 in the case of surfaces of constant width in R^3 by the following two examples.

Example 4.3.1. For the support function $((u) = 2\cos 3u + 3$, the parametrization F of the surface S is defined by

$$F(u,v) = (x(u,v), y(u,v), z(u,v))$$

where

$$x = 2\cos 3u\cos u + 3\cos u + 6\sin 3u\sin u,$$

$$y = 2\sin u\cos 3u\cos v + 3\sin u\cos v - 6\sin 3u\cos v\cos u$$

and

$$z = 2\cos 3u\sin v\sin u + 3\sin u\sin v - 6\sin 3u\cos u\sin v.$$

This parametrization has a singularity. For, By Theorem 4.2.4 with

$a = 2,\ b = 3$ and $n = 3$ we have .

Example 4.3.2. For the support function $((u) = \cos 3u + 10$, the parametrization F of the surface S is defined by

$$F(u,v) = (x(u,v), y(u,v), z(u,v))$$

where

$$x = \cos u \cos 3u + 10 \cos u + 3 \sin 3u \sin u,$$

$$y = \cos 3u \sin u \cos v + 10 \sin u \cos v - 3 \sin 3u \cos u \cos v$$

and

$$z = \cos 3u \sin u \sin v + 10 \sin u \sin v - 3 \sin 3u \cos u \sin v.$$

This parametrization has no singularity. For, By Theorem 4.2.4 with

$a = 1, b = 10$ and $n = 3$ we have .

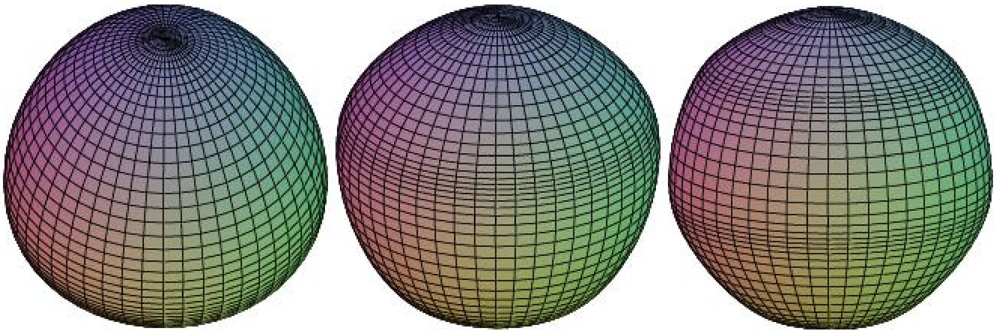
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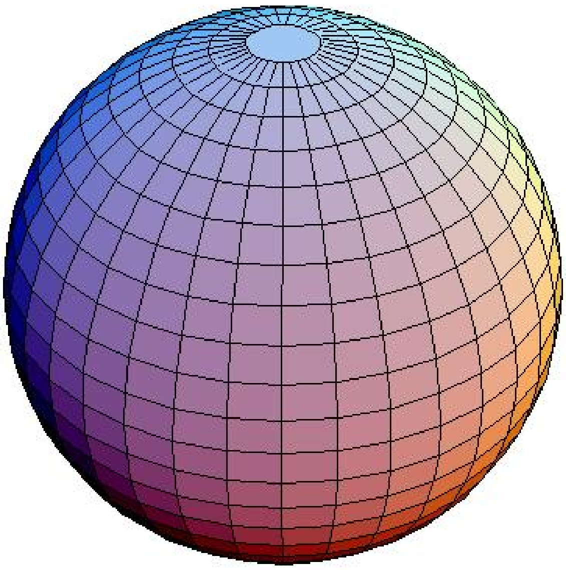
Appendix (I)

Some Figures of Surfaces of Constant Width

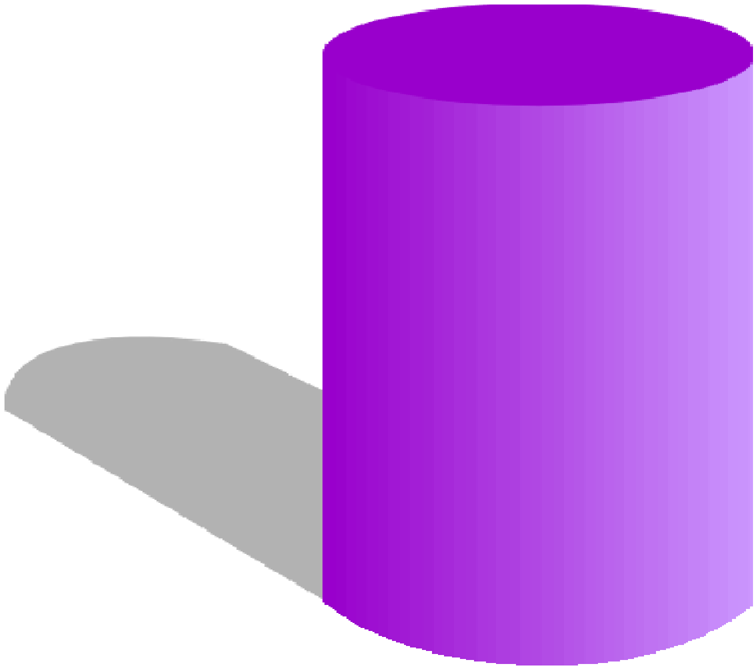


Ovaloid of constant width

.



The sphere is the most obvious example of surfaces of constant width



The cylinder is an example of a surface of constant width

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